# On the Existence of a Solution to the Italian Irregular Immigration Problem:

## A Spectral, Max-Flow and Optimal-Transport Approach

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**Abstract**

We develop an operator-theoretic framework for the Italian irregular immigration problem, a question so far studied mainly on talk shows. Everyone carries a *bureaucratic age*, the time since their last stamp, the residence permit acquires a basic reproduction number, and the irregular stock becomes a Neumann series in a *next-generation operator*. Stricter asylum raises irregularity exactly when returns fall below a threshold that sees every queue at the Questura only through one probability. Buck-passing is a heat equation, anti-smuggling budgets sit on one cut, salience has a mean set by people and a variance set by boats (the *Barge Theorem*), and talk shows implement the power method, whose ground state of outrage sits on a pier. Every electorate has a solution, though no policy brings the irregular stock to zero; for a finite menu with distinct outcomes it is generically unique, and jumps at the policy reversals we call *turnarounds*. The *Excluded Middle Theorem* finds the floor of the irregular stock: every decision ends in a permit or a departure, no permit ends in irregularity, and arrivals of applications stop mattering. A decree in four articles reaches the floor and so solves the problem; voters decide whether it is worth the price, and which price. Each half will dislike one article, and none of them concerns the sea.

# 1 Bureaucratic Age and the Next-Generation Operator

## 1.1 Statuses, hazards and survival

Throughout, $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space. Let

$$\mathbb{S} = \{\mathsf{W}, \mathsf{L}, \mathsf{U}\}$$

be the set of *statuses*:

- $\mathsf{W}$: an application is pending;
- $\mathsf{L}$: a valid residence permit is held;

- $\mathsf{U}$: neither (irregular).

A fourth symbol † denotes exit from the model (return, emigration, naturalization, onward movement, death).

**Definition 1.1** (Bureaucratic age)**.** The *bureaucratic age* $s \geq 0$ of an individual in status $i \in \mathbb{S}$ is the time, in years, elapsed since that individual entered status $i$. It is reset to zero at every change of status.

*Remark* 1.2. Bureaucratic age is the only form of seniority that the Italian administration recognizes without a public competition.

**Definition 1.3** (Transition hazards)**.** For every ordered pair $(i, j)$ with $i \in \mathbb{S}$ and $j \in \mathbb{S} \cup \{\dagger\}$, $j \neq i$, let $\mu_{ij} : [0, \infty) \to [0, \infty)$ be a locally integrable function, the *hazard* of moving from $i$ to $j$ at bureaucratic age $s$. The *total hazard* of $i$ is

$$\mu_i = \sum_{j \neq i} \mu_{ij}.$$

**Definition 1.4** (Survival and mean sojourn)**.** The *survival function* of status $i$ and its *mean sojourn* are

$$S_i(s) = \exp\Big(-\int_0^s \mu_i(\sigma)\, d\sigma\Big), \qquad m_i = \int_0^\infty S_i(s)\, ds. \tag{1}$$

$S_i(s)$ is the probability that a spell in $i$ is still running at bureaucratic age $s$.

**Assumption 1.5** (Finite sojourns)**.** For all $i \in \mathbb{S}$, $m_i < \infty$ and $S_i(s) \to 0$ as $s \to \infty$.

Assumption 1.5 is in force throughout the paper. Nobody waits forever, at least in the model.

Only some transitions are allowed:

- from $\mathsf{W}$, a decision leads to $\mathsf{L}$, to $\mathsf{U}$ (rejection without return) or to † (rejection with return);
- from $\mathsf{L}$, the permit is either lost ($\mathsf{U}$) or the holder leaves (†);
- from $\mathsf{U}$, the individual is either regularized ($\mathsf{L}$) or leaves (†).

Formally $\mu_{\mathsf{LW}} = \mu_{\mathsf{UW}} = 0$: nobody goes back to waiting. On this point alone, the model is more merciful than the administration it describes.

## 1.2 The next-generation operator

A spell in one status ends in another status or in an exit. To count stamps we only need to know where spells end, not when.

**Definition 1.6** (Exit probabilities)**.** For $i \in \mathbb{S}$ and $j \in \mathbb{S} \cup \{\dagger\}$, $j \neq i$, let

$$\pi_{ij} = \int_0^\infty \mu_{ij}(s) S_i(s)\, ds \in [0, 1], \tag{2}$$

the probability that a spell in $i$ ends with a move to $j$.

**Definition 1.7** (Named probabilities)**.** We write

$$p = \pi_{\mathsf{WL}}, \qquad (1-p)(1-r) = \pi_{\mathsf{WU}}, \qquad (1-p)r = \pi_{\mathsf{W}\dagger},$$

which is consistent because $\pi_{\mathsf{WU}} + \pi_{\mathsf{W}\dagger} = 1 - p$ (Lemma 1.9(i) below). This defines:

- the *final recognition probability* $p \in [0,1]$;
- when $p < 1$, the *return probability after rejection* $r \in [0,1]$; when $p = 1$, $r$ plays no role and is fixed arbitrarily in $[0,1]$;
- $\ell = \pi_{\mathsf{LU}}$, the probability of *losing* a permit before leaving;
- $g = \pi_{\mathsf{UL}}$, the probability of being *regularized* before leaving.

**Definition 1.8** (Next-generation operator)**.** The *next-generation operator* is the nonnegative matrix $K \in \mathbb{R}^{\mathbb{S}\times\mathbb{S}}$ with entries $K_{ji} = \pi_{ij}$ for $i \neq j$ and $K_{ii} = 0$. Column $i$ of $K$ lists where a spell started in $i$ sends its holder next (Figure 1). Ordering the statuses as $(\mathsf{W}, \mathsf{L}, \mathsf{U})$,

$$K = \begin{pmatrix} 0 & 0 & 0 \\ p & 0 & g \\ (1-p)(1-r) & \ell & 0 \end{pmatrix}. \tag{3}$$

**Lemma 1.9.** *For the next-generation operator $K$ of* (3)*:*

*(i) For every $i \in \mathbb{S}$, $\sum_{j\neq i} \pi_{ij} = 1$; in particular $\pi_{\mathsf{WL}} + \pi_{\mathsf{WU}} + \pi_{\mathsf{W}\dagger} = 1$, $\ell + \pi_{\mathsf{L}\dagger} = 1$ and $g + \pi_{\mathsf{U}\dagger} = 1$, and each column of $K$ sums to at most 1.*

*(ii) The spectrum of $K$ is $\{0, \sqrt{\ell g}, -\sqrt{\ell g}\}$, so its spectral radius is*

$$\mathcal{R}_0 := \rho(K) = \sqrt{\ell g}.$$

*(iii) If $\ell g < 1$, then $I - K$ is invertible and $(I-K)^{-1} = \sum_{n\geq 0} K^n$ with nonnegative entries.*

*Proof.* **(i)** Since $\frac{d}{ds} S_i = -\mu_i S_i$ and $S_i(s) \to 0$,

$$\sum_{j\neq i} \pi_{ij} = \int_0^\infty \mu_i S_i \, ds = S_i(0) - \lim_{s\to\infty} S_i(s) = 1.$$

Dropping the exit term gives column sums at most one.

**(ii)** The characteristic polynomial of the matrix (3) is $-\lambda(\lambda^2 - \ell g)$.

**(iii)** If $\ell g < 1$ the spectral radius is below one and the Neumann series converges; its terms are nonnegative. □

*Remark* 1.10 (The residence permit reproduces). $\mathcal{R}_0$ is the basic reproduction number of the residence permit: the geometric mean of the probability of losing it and the probability of getting it back. Both events compete with leaving, so $\mathcal{R}_0 < 1$ as soon as a permit holder or an irregular resident has a positive probability of leaving. Below the threshold the expected number of repeat stamps is finite.

The procurement office may nevertheless budget for infinity.

### 1.3 The McKendrick system

Entries come from outside the model: new applications enter $\mathsf{W}$, legal-channel entrants enter $\mathsf{L}$, and direct irregular entries—expired visas, legal procedures that failed—enter $\mathsf{U}$.

**Definition 1.11** (Entry rates)**.** Let $b = (b_{\mathsf{W}}, b_{\mathsf{L}}, b_{\mathsf{U}}) \in \mathbb{R}^3_{\geq 0}$ be the external entry rates (persons per year), with $a := b_{\mathsf{W}}$ the arrival rate of applications.

**Definition 1.12** (Structured population)**.** A *stationary structured population* is a triple of nonnegative densities $n_i \in L^1(0,\infty)$, $n_i(s)$ being the number of individuals of status $i$ and bureaucratic age $s$ per unit of age, which are locally absolutely continuous and satisfy

$$\begin{aligned} n_i'(s) &= -\mu_i(s)\, n_i(s) \qquad (s > 0), \\ n_i(0) &= b_i + \sum_{j \neq i} \int_0^\infty \mu_{ji}(s)\, n_j(s)\, ds. \end{aligned} \tag{4}$$

The *stationary stocks* are

$$N_i = \int_0^\infty n_i(s)\, ds; \qquad \text{in particular} \qquad U^* = N_{\mathsf{U}}.$$

The first equation is transport in bureaucratic age with loss. The boundary condition states that the flux of newcomers into status $i$ equals external entries plus transitions from the other statuses.

**Theorem 1.13** (The Stationary Police Station Theorem (Teorema della Questura Stazionaria))**.** *Assume $\ell g < 1$. System* (4) *has a unique stationary structured population. It is*

$$n_i(s) = J_i S_i(s),$$

*where the* boundary flux $J = (J_{\mathsf{W}}, J_{\mathsf{L}}, J_{\mathsf{U}})$ *is*

$$J = (I - K)^{-1} b = \sum_{n \geq 0} K^n b, \tag{5}$$

*and the stationary stocks are $N_i = m_i J_i$. Explicitly,*

$$U^* = \frac{m_{\mathsf{U}}}{1 - \ell g}\Big[b_{\mathsf{U}} + \ell\, b_{\mathsf{L}} + a\big((1-p)(1-r) + p\,\ell\big)\Big]. \tag{6}$$

*Proof.* **Step 1: transport.** On $(0,\infty)$ the first equation of (4) is a linear ODE whose locally absolutely continuous solutions are exactly

$$n_i(s) = n_i(0) S_i(s).$$

Set $J_i = n_i(0)$. Then $n_i \in L^1$ since $m_i < \infty$.

**Step 2: the boundary condition is a linear system.** The boundary term becomes

$$\int_0^\infty \mu_{ji} n_j \, ds = J_j \pi_{ji} = K_{ij} J_j.$$

The boundary conditions therefore read

$$J = b + KJ,$$

which by Lemma 1.9 has the unique solution $J = (I-K)^{-1} b$. It is nonnegative because the Neumann series has nonnegative terms. Conversely, this $J$ defines a stationary population, and $N_i = J_i \int S_i = m_i J_i$.

**Step 3: the formula for $U^*$.** We have $J_{\mathsf{W}} = a$, and the remaining equations are

$$\begin{aligned} J_{\mathsf{L}} &= b_{\mathsf{L}} + pa + g J_{\mathsf{U}}, \\ J_{\mathsf{U}} &= b_{\mathsf{U}} + (1-p)(1-r)a + \ell J_{\mathsf{L}}. \end{aligned}$$

Substituting the first into the second gives

$$(1 - \ell g) J_{\mathsf{U}} = b_{\mathsf{U}} + (1-p)(1-r)a + \ell(b_{\mathsf{L}} + pa),$$

and (6) follows from $U^* = m_{\mathsf{U}} J_{\mathsf{U}}$. □

*Remark* 1.14 (Generations of stamps). The Neumann series (5) has a literal reading. The term $K^n b$ is the flux of entries into statuses that are a person's $(n+1)$-th status: $b$ is the first stamp, $Kb$ the second, $K^2 b$ the third. The irregular stock is the sum over all generations of stamps.

The factor $(1-\ell g)^{-1}$ in (6) is the expected number of irregular spells of a person who starts one, because each round trip $\mathsf{U} \to \mathsf{L} \to \mathsf{U}$ succeeds with probability $g\ell$.

*Remark* 1.15 (What we do not claim). We do not claim here that time-dependent solutions converge to the stationary one. That requires the theory of positive semigroups for structured populations [9], and we prefer to state only what we prove.

## 1.4 Insensitivity of the return threshold

Formula (6) answers a question that returns in every asylum debate: does a stricter filter on applications reduce irregularity?

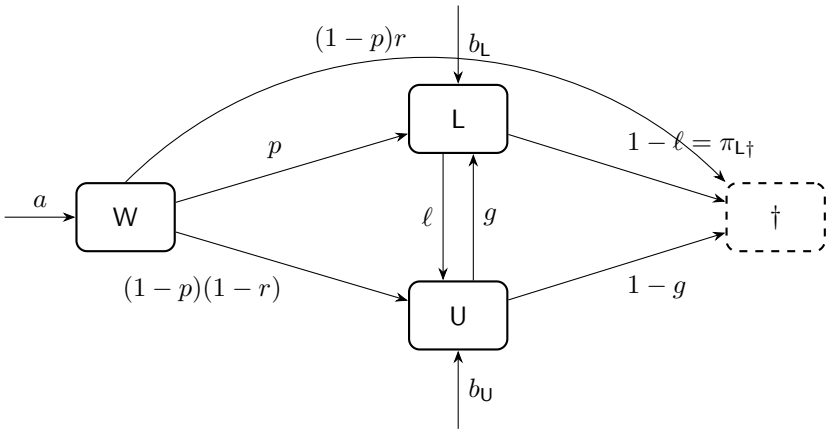



Figure 1: The next-generation operator $K$. External arrows ($a$, $b_{\mathsf{L}}$, $b_{\mathsf{U}}$) are entry rates; all other arrows are transition probabilities, not rates. Every arrow hides a distribution of waiting times, and Theorem 1.16 says that, for the sign of $\partial U^*/\partial p$, they matter only through the probabilities on the arrows. The loop $\mathsf{L} \rightleftarrows \mathsf{U}$ is the residence permit's attempt at reproduction; its success probability is $\mathcal{R}_0^2 = \ell g < 1$.

**Theorem 1.16** (Insensitive return threshold)**.** *Assume $\ell g < 1$ and $a > 0$. Regard $U^*$ in* (6) *as a function of $p$, with $r$, $\ell$, $g$, $m_{\mathsf{U}}$ and $b$ held fixed (changing the decision probabilities out of* W *does not affect hazards out of* L *or* U*). Then*

$$\frac{\partial U^*}{\partial p} = \frac{a\, m_{\mathsf{U}}}{1-\ell g}(r-\bar{r}), \qquad \bar{r} := 1-\ell = \pi_{\mathsf{L}\dagger}. \tag{7}$$

*Hence:*

*(i) a stricter filter (smaller $p$) increases $U^*$ if and only if $r < \bar{r}$;*

*(ii) the threshold $\bar{r}$ is the probability that a permit holder leaves before losing the permit, and it depends on the hazards $\mu_{\mathsf{L}\mathsf{U}}, \mu_{\mathsf{L}\dagger}$ only through this single number;*

*(iii) given $\ell$, it does not depend on $m_{\mathsf{W}}$, $m_{\mathsf{L}}$, $m_{\mathsf{U}}$, on $g$, or on the shape of any sojourn distribution; changing the shape of waiting times matters only insofar as it changes $\ell$.*

*Proof.* In (6) only the term $a((1-p)(1-r)+p\ell)$ depends on $p$, with derivative

$$a\big(\ell-(1-r)\big) = a\big(r-(1-\ell)\big).$$

The prefactor $m_{\mathsf{U}}/(1-\ell g)$ is positive. □

*Remark* 1.17 (Not an artefact of exponentials). With exponential waiting times—permit lost at rate $\delta$, departure at rate $\eta$—one has

$$\ell = \frac{\delta}{\delta+\eta}, \qquad \text{and the threshold reads} \qquad \bar{r} = \frac{\eta}{\delta+\eta}.$$

Theorem 1.16 shows this is not an artefact of exponentials. Within the class of Definition 1.3 and Assumption 1.5, i.e. for finite-mean waiting times generated by locally integrable hazards, the same comparison holds, because only the order of two events matters.

Queueing theory calls such results *insensitivity* [1]. Italy calls them *the same old story.*

## 2 The Buck-Passing Laplacian (Laplaciano dello Scaricabarile)

Files pass through a network of offices, each hoping another desk will absorb it. We model the passing.

**Definition 2.1** (Office network)**.** Let $G = (V, E, w)$ be a finite connected undirected graph with $|V| = n \geq 2$ offices and symmetric weights $w_{xy} = w_{yx} > 0$ on edges $\{x, y\} \in E$ (and $w_{xy} = 0$ otherwise). Here $w_{xy}$ is the rate, per year and per file of difference, at which files are passed between $x$ and $y$.

**Definition 2.2** (Buck-Passing Laplacian)**.** The *degree* of office $x$ and its extremes are

$$d_x = \sum_y w_{xy}, \qquad d_{\max} = \max_x d_x, \qquad d_{\min} = \min_x d_x.$$

With $A = (w_{xy})$ and $D = \operatorname{diag}(d_x)$, the *Buck-Passing Laplacian* is the matrix

$$\mathcal{L} = D - A.$$

**Definition 2.3** (Buck-passing dynamics)**.** The backlog vector $x(t) \in \mathbb{R}^V$ evolves by

$$x' = -\mathcal{L}x. \tag{8}$$

Each office passes files to each neighbour at a rate proportional to how much fuller it is.

Equation (8) is the heat equation on $G$ and the formal definition of *scaricabarile.*

$\mathcal{L}$ is symmetric positive semidefinite, with

$$\langle x, \mathcal{L}x \rangle = \sum_{\{y,z\} \in E} w_{yz}(x_y - x_z)^2.$$

Its eigenvalues are $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$, and $\lambda_1 = 0$ is simple with eigenvector $\mathbf{1}$ because $G$ is connected [3]. The second eigenvalue $\lambda_2$ is the *Fiedler eigenvalue.*

**Theorem 2.4** (The Buck-Passing Theorem (Teorema dello Scaricabarile))**.** *For every initial backlog $x(0)$:*

*(i) the total backlog $\mathbf{1}^\top x(t)$ is constant;*

*(ii) with $\bar{x} = \mathbf{1}^\top x(0)/n$,*

$$\|x(t) - \bar{x}\mathbf{1}\|_2 \leq e^{-\lambda_2 t} \, \|x(0) - \bar{x}\mathbf{1}\|_2.$$

*Proof.* **(i)** $\dfrac{d}{dt}\mathbf{1}^\top x = -\mathbf{1}^\top \mathcal{L} x = -(\mathcal{L}\mathbf{1})^\top x = 0.$

**(ii)** The vector $z = x - \bar{x}\mathbf{1}$ satisfies $z' = -\mathcal{L}z$ and stays orthogonal to $\mathbf{1}$. Expanding $z$ in an orthonormal eigenbasis of $\mathcal{L}$ restricted to $\mathbf{1}^\perp$, each component decays at least like $e^{-\lambda_2 t}$. □

*Remark* 2.5. Buck-passing never reduces the backlog. It spreads the same files more evenly—the one service the administration provides with perfect equality.

How fast it equalizes depends on the narrowest bottleneck in the network.

**Definition 2.6** (Cheeger constant)**.** For $\emptyset \neq Q \subsetneq V$ let

$$\mathrm{vol}(Q) = \sum_{x \in Q} d_x, \qquad w(Q, Q^c) = \sum_{x \in Q,\, y \notin Q} w_{xy}.$$

The *Cheeger constant* of $G$ is

$$h_G = \min\Big\{ \frac{w(Q, Q^c)}{\mathrm{vol}(Q)} : \ \mathrm{vol}(Q) \le \mathrm{vol}(V)/2 \Big\}.$$

**Theorem 2.7** (Cheeger inequality [3])**.** *Let $\mathcal{L}_{\mathrm{norm}} = D^{-1/2}\mathcal{L}D^{-1/2}$ have second eigenvalue $\nu_2$. Then*

$$\frac{h_G^2}{2} \le \nu_2 \le 2h_G.$$

*Moreover $d_{\min}\nu_2 \le \lambda_2 \le d_{\max}\nu_2$, so that $\lambda_2 \ge d_{\min} h_G^2/2$.*

*Proof.* The two-sided bound for $\nu_2$ is the Cheeger inequality for the normalized Laplacian [3].

For the comparison, $\mathcal{L} = D^{1/2}\mathcal{L}_{\mathrm{norm}}D^{1/2}$ is congruent to $\mathcal{L}_{\mathrm{norm}}$. Ostrowski's theorem [6] gives, for every $k$,

$$\lambda_k(\mathcal{L}) = \theta_k\, \lambda_k(\mathcal{L}_{\mathrm{norm}}) \qquad \text{for some } \theta_k \in [d_{\min}, d_{\max}].$$

With $k = 2$ and $\nu_2 \ge 0$, $d_{\min}\nu_2 \le \lambda_2 \le d_{\max}\nu_2$. □

*Remark* 2.8 (The Strait of Messina). Suppose the offices of the Boot split into two halves joined by few edges of small weight. Then the Cheeger constant is small, and so may be $\lambda_2$. When $\lambda_2$ is small, backlogs in the two halves equalize slowly, whatever happens inside each half (Figure 2).

We call such a cut the *Strait of Messina*: a bottleneck with a long literature and no proof of removal.

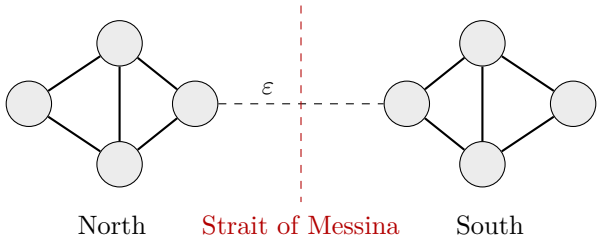



Figure 2: Two well-connected regions joined by a single weak edge of weight $\varepsilon$. The cut $Q = \text{North}$ has $w(Q, Q^c) = \varepsilon$, so $h_G \le \varepsilon/\mathrm{vol}(Q)$ and, by Theorem 2.7, $\nu_2 \le 2h_G \to 0$ as $\varepsilon \to 0$; with the other weights fixed, $d_{\max}$ stays bounded and Ostrowski's bound $\lambda_2 \le d_{\max}\nu_2$ gives $\lambda_2 \to 0$: backlogs equalize within each region quickly and across the Strait slowly.

## 3 Interdiction, or a Measure Supported on Somebody Else's Cut

Now to sea, where routes form a capacitated network and arrivals a maximum flow. The usual question is what happens when one capacity shrinks. A government faces a different one: where to spend a *budget*.

**Definition 3.1** (Route network)**.** Let $(V, E)$ be a finite directed graph with source $s$ and sink $t$ (the Boot), capacities $h \in \mathbb{R}^E_{\ge 0}$, and a partition

$$E = E_{\text{land}} \sqcup E_{\text{sea}},$$

where $E_{\text{sea}}$ is the set of edges entering $t$.

**Definition 3.2** (Rescue floor)**.** A *rescue floor* is a vector $h^\natural$ with $0 \le h^\natural_e \le h_e$ for $e \in E_{\text{sea}}$, encoding the duty of rescue as an admissibility axiom; set $h^\natural_e = 0$ for $e \in E_{\text{land}}$.

**Definition 3.3** (Cuts and maximal flow)**.** A *cut* is a set $S \subset V$ with $s \in S \not\ni t$. Its edge set is $\delta^+(S) = \{(x, y) \in E : x \in S,\ y \notin S\}$, and its capacity is

$$\mathrm{cap}_h(S) = \sum_{e \in \delta^+(S)} h_e.$$

By the max-flow min-cut theorem [5], the maximal flow is

$$F(h) = \min_S \mathrm{cap}_h(S).$$

We assume that reducing any capacity by one person per year costs the same amount, normalized to one, so that budgets are measured in units of capacity.

**Definition 3.4** (Interdiction)**.** An *interdiction* with budget $B \ge 0$ is a vector $z \in \mathbb{R}^E$ with

$$0 \le z_e \le h_e - h^\natural_e, \qquad \sum_e z_e \le B;$$

it replaces $h$ by $h - z$.

**Definition 3.5** (Reducible capacity and interdicted value)**.** For a cut $S$, its *reducible capacity* and the *interdicted value* are

$$R(S) = \sum_{e \in \delta^+(S)} (h_e - h_e^\natural), \qquad F_B = \min\{F(h - z) : z \text{ an interdiction with budget } B\}.$$

**Theorem 3.6** (Interdiction theorem)**.**

$$F_B = \min_S \Big[ \mathrm{cap}_h(S) - \min\big(B, R(S)\big) \Big]. \tag{9}$$

*An optimal interdiction is obtained by choosing a cut $S^\star$ attaining the minimum and spending $\min(B, R(S^\star))$ on edges of $\delta^+(S^\star)$ only, in any way that respects the bounds $z_e \le h_e - h_e^\natural$.*

*Proof.* **Step 1: exchange the minima.**

$$F_B = \min_z \min_S \mathrm{cap}_{h-z}(S) = \min_S \min_z \mathrm{cap}_{h-z}(S),$$

minima over finite sets of cuts and compact sets of interdictions commuting.

**Step 2: a fixed cut.** For fixed $S$,

$$\mathrm{cap}_{h-z}(S) = \mathrm{cap}_h(S) - \sum_{e \in \delta^+(S)} z_e,$$

and the sum is at most $\min(B, R(S))$. This bound is attained by spending the budget on $\delta^+(S)$ up to the bounds, since the bounds on $\delta^+(S)$ sum to $R(S)$. Spending outside $\delta^+(S)$ cannot lower $\mathrm{cap}_{h-z}(S)$. □

**Corollary 3.7** (Externalization)**.** *For the interdicted value $F_B$ of Theorem 3.6:*

*(i) $B \mapsto F_B$ is nonincreasing and piecewise linear, with slope $-1$ or $0$ on each of finitely many intervals.*

*(ii) $F_B = 0$ for some finite $B$ if and only if some cut has $\mathrm{cap}_h(S) = R(S)$, i.e. $h_e^\natural = 0$ on all of $\delta^+(S)$.*

*(iii) If every sea edge has $h_e^\natural > 0$, every such cut lies entirely on land.*

*Proof.* **(i)** Each map

$$\varphi_S(B) = \mathrm{cap}_h(S) - \min(B, R(S))$$

is nonincreasing and piecewise linear, with slope $-1$ on $[0, R(S)]$ and $0$ afterwards. A minimum of finitely many such maps is nonincreasing, piecewise linear, and on each piece coincides with one $\varphi_S$.

**(ii)** $F_B = 0$ requires a cut with $\mathrm{cap}_h(S) - \min(B, R(S)) = 0$, hence $R(S) = \mathrm{cap}_h(S)$; conversely $B = R(S)$ then suffices.

**(iii)** Sea edges with positive floor cannot belong to such a cut. □

*Remark* 3.8 (Memoranda of Understanding). An optimal anti-smuggling budget is therefore a measure supported on a single cut. Under a rescue floor that is strictly positive on every sea edge, the only cuts on which it can drive arrivals to zero are on land; for smaller budgets, the optimal cut may well be at sea.

Diplomacy has a term for a measure supported on a cut of somebody else's land: a Memorandum of Understanding. Memoranda have a half-life for the reason the theorem gives—the minimizing cut depends on $h$, and smugglers, who signed nothing, are free to change $h$.

*Remark* 3.9 (The first unit). $F_B$ need not be convex: budgets can have *increasing* returns. Take two cuts, $S_1$ with capacity 10 and reducible capacity 2, and $S_2$ with capacity 9 and reducible capacity 0. Then

$$F_B = \begin{cases} 9, & B \le 1, \\ 10 - B, & 1 \le B \le 2, \\ 8, & B \ge 2. \end{cases}$$

The first unit of budget buys nothing, because it is spent on a cut that is not yet the bottleneck.

This is the mathematical basis of the sentence *servono più fondi*, which is occasionally true.

## 4 Dublin and the Transport Problem Without Politics

Once ashore, arrivals have to be distributed across the continent. Optimal transport measures what that costs.

**Definition 4.1** (Couplings)**.** Let $(X, d)$ be a Polish metric space (Europe), $\mu, \nu$ Borel probability measures on $X$, and $\Pi(\mu, \nu)$ the set of probability measures on $X \times X$ with marginals $\mu$ and $\nu$. Here $\mu$ is the distribution of landings and $\nu$ the target distribution of reception capacity.

**Definition 4.2** (Wasserstein distance)**.** For $p \ge 1$, the (extended-valued) *Wasserstein distance* is

$$W_p(\mu, \nu) = \inf_{\gamma \in \Pi(\mu,\nu)} \Big( \int d(x, y)^p \, d\gamma \Big)^{1/p}$$

[8]; it is finite when $\mu, \nu$ have finite $p$-th moments.

**Proposition 4.3** (A single port has no politics)**.** *If $\mu = \delta_{x_0}$, then $\Pi(\delta_{x_0}, \nu) = \{\delta_{x_0} \otimes \nu\}$, and*

$$W_p(\delta_{x_0}, \nu)^p = \int_X d(x_0, y)^p \, d\nu(y).$$

*Proof.* Let $\gamma \in \Pi(\delta_{x_0}, \nu)$. Then

$$\gamma((X \setminus \{x_0\}) \times X) = \delta_{x_0}(X \setminus \{x_0\}) = 0,$$

so $\gamma$ is concentrated on $\{x_0\} \times X$. For Borel $B \subset X$, $\gamma(X \times B) = \gamma(\{x_0\} \times B) = \nu(B)$. Thus, on measurable rectangles,

$$\gamma(A \times B) = \mathbf{1}_A(x_0)\nu(B) = (\delta_{x_0} \otimes \nu)(A \times B).$$

Rectangles form a $\pi$-system generating the Borel $\sigma$-algebra of $X \times X$, so $\gamma = \delta_{x_0} \otimes \nu$, and the cost formula follows. □

*Remark* 4.4 (Moment of inertia). When every arrival lands at the same pier $x_0$, the transport problem has exactly one admissible plan, hence exactly one optimal plan. There is nothing to negotiate in the plan itself; disagreement concerns the choice of $\nu$.

The cost of European solidarity is the $p$-th moment of the capacity distribution about the pier—for $p = 2$, its moment of inertia around Lampedusa.

**Definition 4.5** (Dublin plan)**.** The *Dublin plan* assigns every landed person to the state of first entry. It is the coupling

$$\gamma = (\mathrm{id}, \mathrm{id})_{\#}\mu, \qquad \gamma(A) = \mu(\{x : (x, x) \in A\}),$$

the image (push-forward) of $\mu$ under $x \mapsto (x, x)$. It belongs to $\Pi(\mu, \nu)$ only if $\nu = \mu$.

*Remark* 4.6. The Dublin plan is optimal, with zero cost, exactly when capacity is distributed like landings. When $\mu = \delta_{x_0}$, that means locating all of Europe's reception capacity at $x_0$.

The hypothesis remains unverified; the logistics, at least, would be superb.

**Proposition 4.7** (Solidarity as a price)**.** *Each of $k$ states has a share $s_j \geq 0$ of a relocation obligation and a private cost $\kappa_j \geq 0$ per person actually relocated to it; it may instead pay a price $P > 0$ per person not relocated. The cost-minimizing choice of relocations $t_j \in [0, s_j]$ is*

$$t_j = \begin{cases} s_j, & \kappa_j < P, \\ 0, & \kappa_j > P, \\ \text{any } t_j \in [0, s_j], & \kappa_j = P. \end{cases}$$

*Proof.* State $j$ minimizes

$$\kappa_j t_j + P(s_j - t_j),$$

which is affine in $t_j$ with slope $\kappa_j - P$. □

*Remark* 4.8 (Revealed preferences). A fixed price does what communiqués avoid: it reveals preferences. A solidarity mechanism with an opt-out price $P$ sorts states into those whose private cost of one relocated person falls below $P$ and those whose cost lies above it.

For a state with a positive obligation, its choice reveals which side of the price threshold its cost lies on, with equality amounting to indifference; a state with no obligation reveals nothing.

# 5 Salience as an Operator

## 5.1 The Barge Theorem (Teorema del Barcone)

Arrivals do not come one person at a time. They come in boats, a detail the evening news does not ignore.

**Definition 5.1** (Boat process)**.** Let $(\tau_k)_{k\in\mathbb{Z}}$ be the points of a homogeneous Poisson process on $\mathbb{R}$ of intensity $\Lambda > 0$ (landings per year), and let $(B_k)$ be i.i.d. positive random variables (persons per boat), independent of the points, with $\mathbb{E}[B_1^2] < \infty$. The mean flow of persons is

$$m = \Lambda \mathbb{E}[B_1].$$

**Definition 5.2** (Salience)**.** The *salience* is the shot noise

$$Y(t) = \sum_{k:\, \tau_k \le t} B_k \, e^{-\xi(t-\tau_k)},$$

where $\xi > 0$ is the forgetting rate of public attention: each landing makes news proportional to its size, and the news fades.

**Theorem 5.3** (The Barge Theorem (Teorema del Barcone))**.** *For every $t$, $Y(t)$ is almost surely finite, and*

$$\mathbb{E}[Y(t)] = \frac{m}{\xi}, \qquad \mathrm{Var}[Y(t)] = \frac{\Lambda \mathbb{E}[B_1^2]}{2\xi} = \frac{m}{2\xi} \cdot \frac{\mathbb{E}[B_1^2]}{\mathbb{E}[B_1]}.$$

*In particular, for boats of deterministic size $\beta$ carrying a fixed mean flow $m$,*

$$\mathrm{Var}[Y(t)] = \frac{m\beta}{2\xi}.$$

*At a fixed number of people, the variance of salience is proportional to the size of the boat.*

*Proof.* **Step 1: marking.** Let $P_B$ be the law of $B_1$. By the marking theorem [7], the marked points $(\tau_k, B_k)$ form a Poisson process on $\mathbb{R} \times (0,\infty)$ with intensity $\Lambda \, d\tau \otimes P_B$.

**Step 2: Campbell's theorem.** Apply Campbell's theorem [7] to

$$f(\tau, \beta) = \beta e^{-\xi(t-\tau)} \mathbf{1}_{\tau \le t}.$$

The sum $\sum_k f(\tau_k, B_k)$ has mean

$$\int f \, d\Lambda \otimes P_B = \Lambda \mathbb{E}[B_1] \int_0^\infty e^{-\xi u} du = \frac{m}{\xi}$$

(finite, hence $Y(t) < \infty$ a.s.) and variance

$$\int f^2 = \Lambda \mathbb{E}[B_1^2] \int_0^\infty e^{-2\xi u} du = \frac{\Lambda \mathbb{E}[B_1^2]}{2\xi}.$$

**Step 3: deterministic boats.** For $B_1 \equiv \beta$, $\Lambda = m/\beta$. □

*Remark* 5.4 (Television reads the second moment). The mean of salience counts people; its fluctuations count boat sizes. At the same mean passenger flow, barconi of five hundred produce the same average attention as barchini of five, and one hundred times the stationary variance (Figure 3).

Mean passenger flow is a first-moment quantity; television reads the second moment, preferably live.

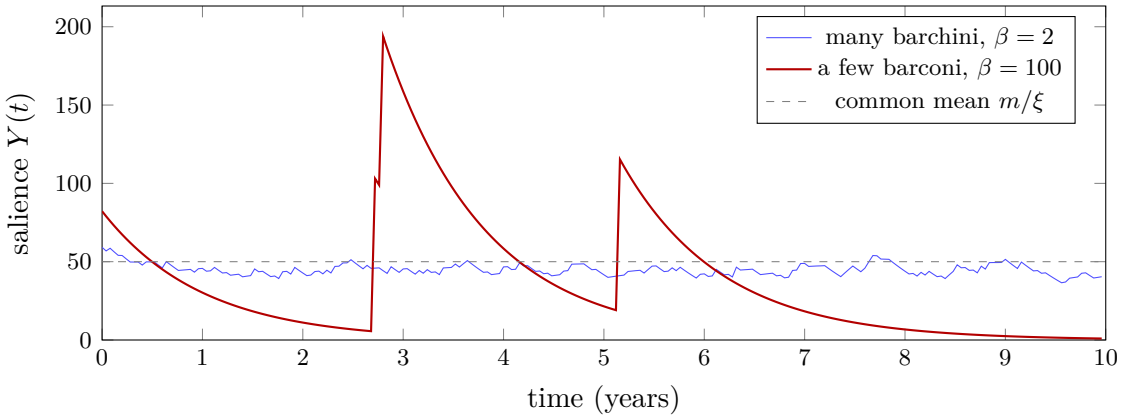



Figure 3: One simulated path of the salience $Y(t)$ for each of two boat processes with the same flow of people, $m = 50$ per year, and forgetting rate $\xi = 1$ (illustrative parameters): $\Lambda = 25$ boats of $\beta = 2$ people per year against $\Lambda = 0.5$ boats of $\beta = 100$. Theorem 5.3 gives both the mean $m/\xi = 50$; the variance $m\beta/(2\xi)$ is 50 for the barchini and 2500 for the barconi. The evening news is drawn in red.

## 5.2 The talk-show operator

Attention also has geography. Let $\mathbb{B} \subset \mathbb{R}^2$ be a compact set (the Boot) with Lebesgue measure, and

$$\mathcal{H} = L^2(\mathbb{B}), \qquad \langle f, h \rangle = \int_{\mathbb{B}} f h.$$

**Definition 5.5** (Talk-show operator)**.** Let $\phi \in L^2(\mathbb{B})$, $\phi \geq 0$, $\phi \neq 0$, be the *landing profile*, concentrated near the points where boats land, and let $k \in L^2(\mathbb{B}\times\mathbb{B})$ be symmetric, $k(x, y) = k(y, x)$, describing the ordinary circulation of news. For $\varepsilon \geq 0$ the *talk-show operator* is the integral operator

$$(Tf)(x) = \phi(x)\langle \phi, f \rangle + \varepsilon \int_{\mathbb{B}} k(x, y) f(y)\, dy.$$

**Definition 5.6** (Attention profile and season)**.** An *attention profile* is a unit vector $f \in \mathcal{H}$, interpreted as an amplitude whose square $|f|^2$ is the density of attention over $\mathbb{B}$. A *season* of television, started from an attention profile $f_0$, is the iteration (defined while $Tf_j \neq 0$)

$$f_{j+1} = \frac{Tf_j}{\|Tf_j\|}.$$

**Lemma 5.7.** *$T$ is a compact self-adjoint operator on $\mathcal{H}$.*

*Proof.* $T$ is the sum of the rank-one operator $\phi \otimes \phi$ and $\varepsilon$ times a Hilbert–Schmidt integral operator with symmetric kernel; both are compact and self-adjoint. □

**Proposition 5.8** (Ground state of outrage)**.** *Let $K_k$ be the integral operator*

$$(K_k f)(x) = \int_{\mathbb{B}} k(x,y) f(y)\, dy,$$

*let $\|K_k\|$ be its operator norm, write $\epsilon_0 = \varepsilon \|K_k\|$, and suppose $2\epsilon_0 < \|\phi\|^2$. Then:*

*(i) the largest eigenvalue $\lambda_1$ of $T$ is simple, and $\lambda_1 \geq \|\phi\|^2 - \epsilon_0$;*

*(ii) the rest of the spectrum of $T$ lies in $[-\epsilon_0, \epsilon_0]$;*

*(iii) the unit eigenvector $e_1$ (with $\langle e_1, \phi \rangle \geq 0$) satisfies*

$$\sin \angle(e_1, \phi/\|\phi\|) \leq \frac{\epsilon_0}{\|\phi\|^2 - \epsilon_0};$$

*(iv) if $\langle f_0, e_1 \rangle \neq 0$, then $\min_{\pm} \|f_j \mp e_1\| = O\big((\epsilon_0/\lambda_1)^j\big)$.*

*Proof.* **Step 1: the unperturbed operator.** The operator $T_0 = \phi\langle \phi, \cdot \rangle$ has spectrum $\{\|\phi\|^2, 0\}$, with eigenvector $\phi/\|\phi\|$ for the nonzero eigenvalue. The perturbation $E = T - T_0$ is self-adjoint with $\|E\| = \epsilon_0$.

**Step 2: a lower bound.** Since $T_0 \geq 0$ and $\|E\| = \epsilon_0$, $T \geq -\epsilon_0$, so the spectrum of $T$ lies in $[-\epsilon_0, \infty)$.

**Step 3: the top eigenvalue, (i) and (ii).** By the min-max principle,

$$\lambda_1(T) = \sup_{\|f\|=1} \langle Tf, f \rangle \geq \frac{\langle T\phi, \phi \rangle}{\|\phi\|^2} \geq \|\phi\|^2 - \epsilon_0,$$

and for every unit $f \perp \phi$,

$$\langle Tf, f \rangle = \langle Ef, f \rangle \leq \epsilon_0.$$

Hence the spectral subspace of $T$ for $(\epsilon_0, \infty)$ has dimension at most one, so $(\epsilon_0, \infty) \cap \sigma(T) = \{\lambda_1\}$ with $\lambda_1$ simple, and the rest of the spectrum lies in $[-\epsilon_0, \epsilon_0]$.

**Step 4: the angle, (iii).** The Davis–Kahan $\sin\theta$ theorem [4], applied with the eigenvalue $\lambda_1$ of $T$ separated by at least $\lambda_1 - 0 \geq \|\phi\|^2 - \epsilon_0$ from the rest of the spectrum of $T_0$, gives the angle bound.

**Step 5: the iteration, (iv).** Expand $f_0$ in an orthonormal eigenbasis of the compact self-adjoint operator $T$:

$$T^j f_0 = \lambda_1^j \langle f_0, e_1 \rangle e_1 + R_j, \qquad \|R_j\| \leq \epsilon_0^j \|f_0\|,$$

and normalize. □

*Remark* 5.9 (The power method with studio lighting). Under the small-perturbation hypothesis of Proposition 5.8, repeated broadcasting is the power method with studio lighting. Provided the initial distribution of attention over the Boot is not exactly orthogonal to the ground state, a season of talk shows converges geometrically to a profile within angle $O(\varepsilon)$ of the landing profile: the ground state of outrage sits on a pier.

Ordinary news ($\varepsilon > 0$) can rotate it by an angle of order $\varepsilon$, now and then toward the rest of the country—which is how the rest of the country finds out it exists.

# 6 The Turnaround Theorem (Teorema del Ribaltone)

Choice, at last. A government picks a policy $u$ in a set $\mathcal{U}$ to minimize a weighted sum of objectives. Existence of a minimizer for every weight vector is standard (Theorem 6.4 below); the harder question is what happens when the weights move.

**Definition 6.1** (Policies and objectives)**.** Let $\mathcal{U}$ be a nonempty compact metric space of policies and

$$\mathbf{f} = (f_1, \dots, f_m) : \mathcal{U} \to \mathbb{R}^m$$

continuous, $m \geq 2$, the normalized objectives (for instance irregular stock, budget, humanitarian cost).

**Definition 6.2** (Electorate)**.** The *electorate* is a weight vector $w$ in the simplex

$$\Delta = \Big\{ w \in \mathbb{R}^m_{\geq 0} : \textstyle\sum w_j = 1 \Big\}.$$

**Definition 6.3** (Value and solutions)**.** For $w \in \Delta$ let

$$\Phi_w = w \cdot \mathbf{f}, \qquad V(w) = \min_{u \in \mathcal{U}} \Phi_w(u), \qquad \mathcal{S}(w) = \arg\min_{u \in \mathcal{U}} \Phi_w(u),$$

the objective, the *value*, and the *solution correspondence*.

**Theorem 6.4** (Berge, and concavity)**.** *$\mathcal{S}$ has nonempty compact values and is upper hemicontinuous on $\Delta$; $V$ is continuous and concave on $\Delta$.*

*Proof.* $(w, u) \mapsto w \cdot \mathbf{f}(u)$ is jointly continuous and the constraint set $\mathcal{U}$ is compact and does not depend on $w$. Berge's maximum theorem [2] gives nonempty compact values, upper hemicontinuity of $\mathcal{S}$ and continuity of $V$.

For each $u$, $w \mapsto w \cdot \mathbf{f}(u)$ is linear, and a pointwise minimum of linear functions is concave. □

*Remark* 6.5 (Coalitions). Concavity of $V$ reads politically as follows: for $\lambda \in [0, 1]$,

$$V(\lambda w + (1 - \lambda) w') \geq \lambda V(w) + (1 - \lambda) V(w').$$

A coalition whose electorate mixes two electorates faces a problem at least as hard as the average of the problems its members would face alone.

Upper hemicontinuity rules out one surprise only: small swings cannot make new policies appear from nowhere. Policies disappearing is another matter, and it is not ruled out.

**Definition 6.6** (Turnaround)**.** An electorate $w^0 \in \Delta$ is a *turnaround* if there are sequences $w_n \to w^0$ and $w'_n \to w^0$ in $\Delta$ and policies $u \neq u'$ in $\mathcal{U}$ with $\mathcal{S}(w_n) = \{u\}$ and $\mathcal{S}(w'_n) = \{u'\}$ for all $n$.

**Theorem 6.7** (The Turnaround Theorem (Teorema del Ribaltone))**.** *Let $\mathcal{U}$ be finite and $\mathbf{f}$ injective. Then:*

*(i) The set $\Delta_{\text{tie}} = \{w \in \Delta : |\mathcal{S}(w)| \geq 2\}$ is contained in a finite union of relatively closed subsets of $\Delta$ of dimension at most $m-2$. On its complement, which is relatively open, dense and of full $(m-1)$-dimensional measure in $\Delta$, $\mathcal{S}$ is single-valued and locally constant.*

*(ii) $V$ is piecewise linear, linear in a relative neighbourhood of every $w$ with $|\mathcal{S}(w)| = 1$, and not differentiable along $\Delta$ at any relatively interior $w$ with $|\mathcal{S}(w)| \geq 2$.*

*(iii) Every $w^0$ in the relative interior of $\Delta$ with $|\mathcal{S}(w^0)| \geq 2$ is a turnaround. In particular $\mathcal{S}$ is not lower hemicontinuous at $w^0$.*

*Proof.* Let

$$D_0 = \Big\{v \in \mathbb{R}^m : \textstyle\sum_j v_j = 0\Big\},$$

the direction space of $\Delta$, and $P$ the orthogonal projection onto $D_0$.

**(i) Ties are rare.** For $u \neq u'$, $d = \mathbf{f}(u) - \mathbf{f}(u') \neq 0$, and a tie requires $w \cdot d = 0$.

- If $d = c\mathbf{1}$, then $w \cdot d = c \neq 0$ on $\Delta$, so no tie occurs.
- Otherwise $Pd \neq 0$ and $\{w \in \Delta : w \cdot d = 0\}$ is the intersection of $\Delta$ with an affine hyperplane transversal to its affine hull, a closed set of dimension at most $m-2$.

There are finitely many pairs, so $\Delta_{\text{tie}}$ lies in a finite union of such sets, which is closed, nowhere dense and of measure zero in $\Delta$. At every $w$ with a unique minimizer, since the finitely many functions $w \mapsto w \cdot \mathbf{f}(u)$ are continuous, the strict inequalities that make it unique persist in a neighbourhood. Hence $\Delta \setminus \Delta_{\text{tie}}$ is relatively open and $\mathcal{S}$ is locally constant on it; it contains the complement of the union, so it is dense and of full measure.

**(ii) $V$ is piecewise linear.** $V$ is a minimum of finitely many linear functions; near a $w$ with unique minimizer $u$, by (i),

$$V(w') = w' \cdot \mathbf{f}(u).$$

Non-differentiability at interior ties is proved at the end of (iii).

**(iii) Interior ties are turnarounds.** Let $\mathcal{T} = \mathcal{S}(w^0)$, $|\mathcal{T}| \geq 2$.

*Step A: tied policies have distinct projections.* For $v \neq v'$ in $\mathcal{T}$, $P\mathbf{f}(v) \neq P\mathbf{f}(v')$. Otherwise $\mathbf{f}(v) - \mathbf{f}(v') = c\mathbf{1}$ with $c \neq 0$ by injectivity, and then

$$w^0 \cdot \mathbf{f}(v) - w^0 \cdot \mathbf{f}(v') = c \neq 0,$$

contradicting that both minimize $\Phi_{w^0}$.

*Step B: two exposed vertices.* So the finite set $\{P\mathbf{f}(v) : v \in \mathcal{T}\} \subset D_0$ has at least two points, and its convex hull has at least two distinct vertices $P\mathbf{f}(\tilde{u}) \neq P\mathbf{f}(\tilde{u}')$. Every vertex of a polytope is exposed: there is $e \in D_0$ with

$$e \cdot P\mathbf{f}(\tilde{u}) < e \cdot P\mathbf{f}(v) \qquad \text{for all } v \in \mathcal{T} \setminus \{\tilde{u}\},$$

and likewise $e'$ for $\tilde{u}'$. Since $e \in D_0$, $e \cdot P\mathbf{f}(v) = e \cdot \mathbf{f}(v)$.

*Step C: small swings select one policy.* Because $w^0$ is relatively interior, $w^0 + te \in \Delta$ for small $t > 0$. For $v \in \mathcal{T}$,

$$\Phi_{w^0+te}(v) = V(w^0) + t\, e \cdot \mathbf{f}(v),$$

uniquely minimized over $\mathcal{T}$ by $\tilde{u}$. For $v \notin \mathcal{T}$, $\Phi_{w^0}(v) \geq V(w^0) + c_0$ for some gap $c_0 > 0$, so for $t$ small enough these remain strictly larger. Hence $\mathcal{S}(w^0 + te) = \{\tilde{u}\}$ for all small $t > 0$, and similarly $\mathcal{S}(w^0 + te') = \{\tilde{u}'\}$. The sequences $w_n = w^0 + e/n$ and $w'_n = w^0 + e'/n$ (for large $n$) show that $w^0$ is a turnaround.

*Step D: no lower hemicontinuity.* If $\mathcal{S}$ were lower hemicontinuous at $w^0$, $\tilde{u}' \in \mathcal{S}(w^0)$ would be a limit of points of $\mathcal{S}(w_n) = \{\tilde{u}\}$, impossible since $\tilde{u}' \neq \tilde{u}$ and $\mathcal{U}$ is finite (discrete).

*Step E: a kink.* Finally, for small $t > 0$ the same argument gives

$$V(w^0 + te) = V(w^0) + t \min_{v \in \mathcal{T}} e \cdot \mathbf{f}(v),$$
$$V(w^0 - te) = V(w^0) - t \max_{v \in \mathcal{T}} e \cdot \mathbf{f}(v).$$

The two one-sided slopes along $e$ differ because $e \cdot \mathbf{f}$ is not constant on $\mathcal{T}$, so $V$ is not differentiable at $w^0$ along $\Delta$. □

*Remark* 6.8 (Changing policy without changing one's mind). The theorem says that, with a finite menu of policies and injective objectives, the electorate almost never ties, and when it ties in the relative interior of $\Delta$, an arbitrarily small swing on one side selects one policy and on the other side another one. In the relative interior of $\Delta$, the kinks of the concave value function $V$ are precisely the places where this happens. Such a point is a *turnaround* (Figure 4).

Every turnaround lies in $\Delta_{\text{tie}}$: $\mathcal{S}$ has closed graph (compact values and upper hemicontinuity, Theorem 6.4), so the policies $u$ and $u'$ of the definition both belong to $\mathcal{S}(w^0)$. Turnarounds are therefore rare and nowhere dense, and an electorate sitting exactly on one can change policy without changing its mind. Whether any turnaround exists at all depends on whether the objectives conflict: if one policy is best at everything, there is none.

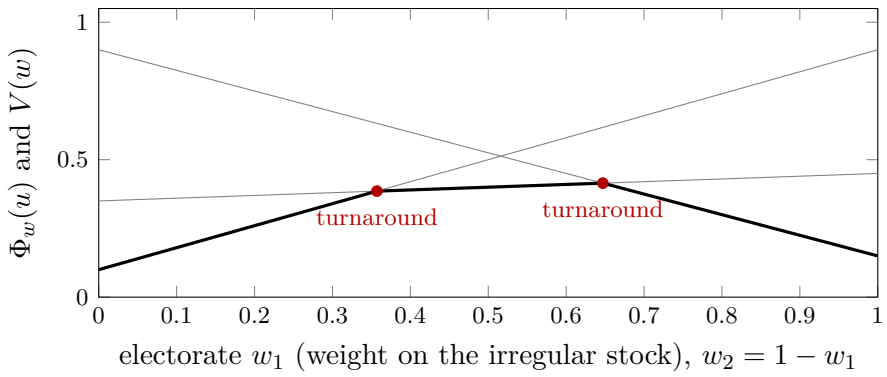



Figure 4: Three hypothetical policies with objective vectors $(0.15, 0.9)$, $(0.45, 0.35)$, $(0.9, 0.1)$ for $m = 2$. Each thin line is $\Phi_w(u)$; the thick lower envelope is the concave value $V$. Its two kinks, at $w_1 = 5/14 \approx 0.357$ and $w_1 = 11/17 \approx 0.647$, are the turnarounds: an arbitrarily small swing across either of them changes the optimal policy, while $V$ itself moves continuously. Concavity means that coalitions face problems at least as hard as the average of their members'.

# 7 The Excluded Middle Theorem (Teorema del Terzo Escluso)

The previous sections proved what operators cannot do. We now ask how low the stationary irregular stock (6) can go, and what a policy must look like to get there.

Throughout this section a policy acts on the decision and permit probabilities

$$(p, r, \ell, g) \in [0, 1]^4, \qquad \ell g < 1,$$

while the external entry rates $b$ (including the arrivals $a = b_{\mathsf{W}}$) and the mean irregular sojourn $m_{\mathsf{U}}$ are held fixed.

**Definition 7.1** (Excluded middle)**.** A policy satisfies the *excluded middle* if

$$\pi_{\mathsf{WU}} = (1 - p)(1 - r) = 0 \qquad \text{and} \qquad \pi_{\mathsf{LU}} = \ell = 0.$$

In words: every decision ends either in a permit ($\mathsf{L}$) or in a departure ($\dagger$), never in irregularity, and no permit ends in irregularity.

**Theorem 7.2** (The Excluded Middle Theorem (Teorema del Terzo Escluso))**.** *Assume $a > 0$, $b_{\mathsf{L}} > 0$ and $b_{\mathsf{U}} > 0$. Then:*

*(i) $U^* \geq m_{\mathsf{U}} b_{\mathsf{U}}$ for every policy, with equality if and only if the policy satisfies the excluded middle;*

*(ii) on the set of policies satisfying the excluded middle, $U^* = m_{\mathsf{U}} b_{\mathsf{U}}$ does not depend on $a$, $p$, $r$ or $g$;*

*(iii) if $\ell = 0$ and $r < 1$, then $\partial U^* / \partial p = -a\, m_{\mathsf{U}}(1 - r) < 0$.*

*Proof.* **Step 1: rewriting $U^*$.** By (6),

$$U^* = \frac{m_{\mathsf{U}}\Gamma}{1-\ell g}, \qquad \Gamma = b_{\mathsf{U}} + \ell\,(b_{\mathsf{L}} + pa) + a(1-p)(1-r). \tag{10}$$

**Step 2: the floor.** The last two terms of $\Gamma$ are nonnegative, so $\Gamma \geq b_{\mathsf{U}} > 0$, and $(1-\ell g)^{-1} \geq 1$; hence $U^* \geq m_{\mathsf{U}} b_{\mathsf{U}}$.

**Step 3: the excluded middle reaches the floor, (ii).** If the excluded middle holds, then $\ell = 0$ gives $1 - \ell g = 1$ and removes the second term of $\Gamma$, and $(1-p)(1-r) = 0$ removes the third, so $U^* = m_{\mathsf{U}} b_{\mathsf{U}}$, which proves (ii).

**Step 4: only the excluded middle reaches the floor.** Conversely, suppose $U^* = m_{\mathsf{U}} b_{\mathsf{U}}$. Since $m_{\mathsf{U}} = \int_0^\infty S_{\mathsf{U}} > 0$ (the survival function is continuous with $S_{\mathsf{U}}(0) = 1$), $\Gamma/(1-\ell g) = b_{\mathsf{U}}$. As $\Gamma \geq b_{\mathsf{U}}$ and $(1-\ell g)^{-1} \geq 1$, this forces $\Gamma = b_{\mathsf{U}}$, i.e.

$$\ell\,(b_{\mathsf{L}} + pa) = 0 \qquad \text{and} \qquad a(1-p)(1-r) = 0.$$

Since $b_{\mathsf{L}} > 0$ and $a > 0$, $\ell = 0$ and $(1-p)(1-r) = 0$.

**Step 5: (iii).** This is Theorem 1.16 with $\ell = 0$, i.e. $\bar{r} = 1$. □

*Remark* 7.3 (Tertium non datur). In logic, the law of the excluded middle says that a proposition is either true or false, with no third value. A decision of the Questura has three possible outcomes, and Theorem 7.2 says that the third one must be empty: a rejection ends in a departure or, failing that, it is not a rejection.

The middle that is excluded is also political. The policy that rejects many, returns few, and lets permits lapse at renewal is not a compromise between two extremes: it is the policy that pays both nonnegative terms of $\Gamma$ in (10).

Part (iii) completes the picture: once permits stop lapsing, the return threshold of Theorem 1.16 equals one, and any strictness short of returning every rejected applicant increases the irregular stock.

**Corollary 7.4** (Return or permit)**.** *Take a policy with $\ell = 0$, and modify it by the rule that a file which would close with a rejection and no return closes with a permit instead. In terms of Definition 1.3,*

$$\hat{\mu}_{\mathsf{WL}} = \mu_{\mathsf{WL}} + \mu_{\mathsf{WU}}, \qquad \hat{\mu}_{\mathsf{WU}} = 0, \qquad \hat{\mu}_{\mathsf{W}\dagger} = \mu_{\mathsf{W}\dagger}.$$

*Then*

$$\hat{\pi}_{\mathsf{WL}} = p + (1-p)(1-r), \qquad \hat{\pi}_{\mathsf{W}\dagger} = (1-p)r, \qquad \hat{\pi}_{\mathsf{WU}} = 0,$$

*so the modified policy satisfies the excluded middle and, under the assumptions of Theorem 7.2, $U^* = m_{\mathsf{U}} b_{\mathsf{U}}$. The departure probability of an applicant and the mean waiting time $m_{\mathsf{W}}$ are unchanged, and the stationary flux into $\mathsf{L}$ increases by $(1-g)(1-p)(1-r)\,a \geq 0$.*

*Proof.* **Probabilities.** The modified hazards are locally integrable and have the same total $\mu_{\mathsf{W}}$, hence the same $S_{\mathsf{W}}$ and $m_{\mathsf{W}}$. Integrating them against $S_{\mathsf{W}}$ gives the three probabilities, so $\hat{\pi}_{\mathsf{WU}} = 0$, and with $\ell = 0$ Definition 7.1 holds and Theorem 7.2(i) applies.

**Flux.** From the proof of Theorem 1.13 with $\ell = 0$,

$$J_{\mathsf{U}} = b_{\mathsf{U}} + \pi_{\mathsf{WU}} a, \qquad J_{\mathsf{L}} = b_{\mathsf{L}} + \pi_{\mathsf{WL}} a + g J_{\mathsf{U}},$$

so the change in $J_{\mathsf{L}}$ is $(1-p)(1-r)a - g(1-p)(1-r)a$. □

*Remark* 7.5 (Relabelling). The rule does not deport anyone, and it does not make anyone wait longer. It relabels: the rejected applicants whom the State fails to return stop being counted in $\mathsf{U}$ and start being counted in $\mathsf{L}$, where, among other things, they can be asked to pay contributions.

*Remark* 7.6 (A decree in four articles). Theorem 7.2 and Corollary 7.4 translate into the following text, which we submit to the Gazzetta Ufficiale with the usual expectations.

**Art. 1.** Every application is decided in finite expected time. This is the assumption $m_{\mathsf{W}} < \infty$ of Assumption 1.5: the decree merely asks the administration to satisfy a hypothesis of the model.

**Art. 2.** Rejected applicants are returned. The return probability $r$ is raised through readmission agreements with the countries of origin, since in practice $r$ is the probability that the country of origin answers the phone.

**Art. 3.** A rejected applicant whom the State has not returned when the file closes receives a permit. The State's failure to return becomes the State's permit, not the applicant's irregularity (Corollary 7.4).

**Art. 4.** A permit ends in a renewal or a departure, never in irregularity: renewal is automatic, the loss of a job does not void the permit, and a permit that is revoked ends with an actual return. Hence $\ell = 0$. Permits may still end; what the article forbids is ending in $\mathsf{U}$.

*Remark* 7.7 (Reading the decree). Articles 3 and 4 alone already reach the floor $m_{\mathsf{U}} b_{\mathsf{U}}$; Article 2 decides how many of the rejected leave and therefore how large the legal stock becomes. Without Article 3, Article 2 must succeed every time ($r = 1$) as soon as anyone is rejected; without Article 2, Article 3 is a permit for everyone who stays.

Each half of the electorate will find one of the two articles unacceptable, which is how one recognizes that the theorem is not taking sides.

Nothing in the decree concerns the sea, and by Theorem 7.2(ii) nothing needs to: under the excluded middle the irregular stock does not depend on the arrival rate $a$ of applications, so

a measure whose only effect is on $a$ may change the number of applicants and permit holders and the associated costs, but not $U^*$. Measures that change $b_{\mathsf{U}}$ or $m_{\mathsf{U}}$ move the floor itself; the model does not say which, if any, sea measures do so.

*Remark* 7.8 (What the theorem does not say). Arrivals are exogenous in the model. If Article 3 attracted more applications, with $b_{\mathsf{U}}$ and $m_{\mathsf{U}}$ unchanged, Theorem 7.2(ii) says that the irregular stock would not change; $N_{\mathsf{W}} = m_{\mathsf{W}} a$ would grow and $N_{\mathsf{L}}$ would not decrease. What happens to the other objectives of Section 6 depends on what they are, which is where the electorate is entitled to disagree.

The theorem also leaves $m_{\mathsf{U}} b_{\mathsf{U}}$ in place: it is generated by people who enter $\mathsf{U}$ directly, without an application, and stay there for a mean time $m_{\mathsf{U}}$ before leaving or being regularized. No decision about applications reaches a flow that never applied.

# 8 Main Theorem

We can now assemble the pieces.

**Definition 8.1** (Classical and weak solutions)**.** A *classical solution* is a policy with $U^* = 0$. A *weak solution* for the electorate $w$ is an element of $\mathcal{S}(w)$.

**Theorem 8.2** (Main Theorem)**.** *Assume:*

*(a) the policy set $\mathcal{U}$ is a compact metric space and the objectives are continuous;*

*(b) $f_1 = U^*/U_{\mathrm{ref}}$ for a reference scale $U_{\mathrm{ref}} > 0$, where for every policy $U^*$ is given by Theorem 1.13 with $\ell g < 1$;*

*(c) there are constants $b_0, m_0 > 0$ with $b_{\mathsf{U}} \geq b_0$ and $m_{\mathsf{U}} \geq m_0$ for every policy.*

*Then:*

*(i) $U^* \geq m_0 b_0 > 0$ for every policy $u \in \mathcal{U}$, so the problem has no classical solution;*

*(ii) for every electorate $w$ the set of weak solutions $\mathcal{S}(w)$ is nonempty and compact, and $\mathcal{S}$ is upper hemicontinuous;*

*(iii) the value $V$ is concave in the electorate;*

*(iv) if the menu of policies is finite and the objective map injective, the solution is unique for almost every electorate, and every relatively interior electorate with more than one solution is a turnaround.*

*Proof.* **(i)** From (6), all terms in the bracket are nonnegative and $(1-\ell g)^{-1} \geq 1$, so

$$U^* \geq m_{\mathsf{U}} b_{\mathsf{U}} \geq m_0 b_0.$$

**(ii)–(iii)** These are Theorem 6.4.

**(iv)** This is Theorem 6.7. □

**Corollary 8.3** (The Solution)**.** *In the setting of Theorem 8.2, suppose in addition that $a$, $b$ and $m_{\mathsf{U}}$ are the same for every policy, with $a > 0$ and $b_{\mathsf{L}} > 0$, and that $\mathcal{U}$ contains a policy satisfying the excluded middle. Then the solutions of the irregular immigration problem proper, i.e. the weak solutions for the electorate $w = e_1 = (1, 0, \ldots, 0)$, the vertex of $\Delta$ that weighs only the irregular stock, are exactly the policies in $\mathcal{U}$ satisfying the excluded middle, and they attain the floor $U^* = m_{\mathsf{U}} b_{\mathsf{U}}$ of Theorem 7.2(i).*

*Proof.* $\mathcal{S}(e_1) = \arg\min_{\mathcal{U}} U^*$, and $b_{\mathsf{U}} \geq b_0 > 0$. By Theorem 7.2(i), $U^* \geq m_{\mathsf{U}} b_{\mathsf{U}}$ on $\mathcal{U}$, with equality exactly on the policies satisfying the excluded middle, which by assumption exist in $\mathcal{U}$. □

*Remark* 8.4 (Who pays). An electorate that weighs other objectives may select a policy above the floor; it is then buying something else with irregular stock, and formula (10) with Theorem 7.2 tells it how much.

The model therefore says which policies solve the problem, and leaves to the electorate the question of whether the solution is worth its price, and which price: through Article 2 of the decree (Remark 7.6), more returns or more permits. With a finite menu and injective objectives, an arbitrarily small swing across a turnaround may change the price.

Whether the country also changes government is outside the model, though not, historically, outside the realm of possibility.

# 9 Conclusion

Waiting times are described by hazards, transitions by operators, populations by densities. On those terms:

- the irregular stock is a Neumann series in the next-generation operator of the residence permit (Theorem 1.13);
- given the probability of leaving before losing the permit, the return threshold is insensitive to the distribution of waiting times (Theorem 1.16);
- buck-passing conserves backlog and equalizes it at the speed of the Fiedler eigenvalue, slowed by the Strait of Messina (Theorems 2.4 and 2.7);
- an optimal anti-smuggling budget is supported on a single cut; under a rescue floor that is strictly positive on every sea edge, only land cuts can be driven to zero, and the first units of budget may buy nothing (Theorem 3.6, Corollary 3.7);

- transport out of a single pier has one plan and no politics, while solidarity at a fixed price reveals what capitals will not say (Propositions 4.3 and 4.7);
- salience has a mean set by people and a variance set by boats (Theorem 5.3), and talk shows implement the power method (Proposition 5.8);
- the value of the problem is a concave function of the electorate whose interior kinks, for a finite menu with injective objectives, are turnarounds (Theorems 6.4 and 6.7).

The Main Theorem then states the conclusion directly: under its assumptions there is no classical solution, but for every electorate the set of weak solutions is nonempty and compact, and, for a finite menu of policies with injective objectives, the solution is unique for almost every electorate, while every relatively interior electorate with more than one solution is a turnaround (Theorem 8.2).

The Excluded Middle Theorem then says what the solution is: the irregular stock reaches its floor if and only if the middle is excluded, after which it no longer depends on the arrival rate of applications (Theorem 7.2, Corollary 8.3).

In one sentence: every *no* ends with a ticket or becomes a *yes*, and every *yes* stays a *yes* or ends with a ticket. Who votes decides whether that is worth paying for, and how many of the tickets are bought.

Future work will address the semigroup convergence of the McKendrick system, the spectral gap of the Conferenza Stato-Regioni, and whether the bridge over the Strait of Messina can be completed within one administrative mixing time $1/\lambda_2$.

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